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Generic Distributions and Lie Algebras of Vector Fields

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1. INTRODUCTION

Distributions, that is, subbundles of the tangent bundle, are generalizations of line fields. It has proven to be very useful to find a frame of vector fields for a given distribution (i.e., the pointwise values of the vector fields form a basis for the distribution) which generates a finite dimensional nilpotent Lie algebra. (For applications to control theory see Hermes, Lundell, and Sullivan [1], Grayson and Grossman [2], and Lafferriere and Sussmann [3]. For applications to hypoelliptic operators see Folland and Stein [4], and Rothschild and Stein [5].) When does such a frame exist? More generally, when does a distribution admit a frame of vector fields which generates a finite dimensional Lie algebra?

The purpose of this note is to prove that the generic rank r distribution on an n dimensional manifold does not admit such a frame, provided

$$r(n-r) > n. \tag{1}$$

We will prove

THEOREM 1. *Let $\mathscr{D}(r,n)$ denote the space of germs at 0 of rank r distributions on $\mathbf{R}^n$. Put the Whitney topology on $\mathscr{D}(r,n)$. (See Section 3 for more on this topology.) Let $\Sigma \subset \mathscr{D}(r,n)$ denote the subset consisting of those distributions which admit some frame which generates a finite dimensional Lie algebra. Suppose that $r(n-r) > n$. Then the complement of Σ is the countable intersection of open dense sets. In particular this complement is dense.*

A special case of this theorem is stated without proof by Gershkovich and Vershik [6] and the present paper was partly inspired by reading their very nice paper.

The only cases excluded by inequality (1) are $r=1$ (vector fields), $r=n-1$, and $r=2$, $n=4$. In each of these cases the generic distribution does admit a frame which generates a finite dimensional algebra. Not only,

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that but in these cases the generic distribution is stable in the usual sense of singularity theory (Arnold *et al.* [8], Golubitsky and Guillemin [7]). For $r = 1$ this fact is called the "straightening out lemma." For $r = n - 1$ this is the Darboux theorem and the algebra generated is the Heisenberg algebra provided n is odd. In the special case $(r, n) = (2, 4)$ the algebra generated is sometimes called the Engel algebra. The theorem for this case is attributed to Engel. See Gardner [9] and Gershkovich and Vershik [6].

2. Heuristic Proof: How to Count

The counting proof which we are about to give contains the main idea behind Theorem 1. A careful proof is given later.

A distribution on $\mathbf{R}^n$ is an assigment of r-dimensional subspace of $\mathbf{R}^n$ to each point of $\mathbf{R}^n$. Thus it is a map from $\mathbf{R}^n$ to $G_{r,n}$ the Grassmannian of r-planes in n-space. $G_{r,n}$ is a manifold of dimension $r(n-r)$ so we need this many functions (of n variables) in order to specify a distribution. Two distributions are equivalent if there is a diffeomorphism of $\mathbf{R}^n$ taking one to the other and n functions are required to sepcify such a diffeomorphism. Thus $r(n-r) - n$ functions of n variables are required to specify a distribution up to diffeomorphism. So when the inequality (1) holds the space of rank r distributions on $\mathbf{R}^n$ up to diffeomorphism is parameterized by a *function space* and as such is infinite dimensional. So the orbit of any single distribution under the action of the diffeomorphism group has infinite codimension. This is the crucial fact.

The set of Lie algebras (up to isomorphism) whose dimension is N forms a finite dimensional real algebraic variety and up to some additional finite dimensional set of parameters this variety parameterizes, again up to diffeomorphism, the space Σ_N of distributions which admit a frame which generates a Lie algebra dimension N. Thus Σ_N is a finite dimensional subvariety of an infinite dimensional variety and as such its complement is open and dense. The set Σ of the theorem is the union of the Σ_N and so its complement is the countable intersection of open dense sets. Such an intersection is dense since the space of distributions forms a Baire space with respect to the Whitney topology (see, for instance, Guillemin and Golubitsky [7]) and we are done.

3. Ingredients of the Proof

If $U \subset \mathbf{R}^n$ is a neighborhood of 0 let $\mathscr{D}(r, n; U)$ denote the space of smooth rank r distributions defined on U. As in the preceding section, we can and will identify $\mathscr{D}(r, n; U)$ with the space of smooth maps from U to the Grassmannian $G_{r,n}$ of r-planes in n-space. Put the Whitney smooth topology on this space of maps. Thus a sequence σ_i converges in $\mathscr{D}(r, n; U)$ if it converges with respect to the C^k topology for all k. If $V \subset U$ we obtain a continuous open map $\mathscr{D}(r, n; U) \to \mathscr{D}(r, n; V)$ by restriction. (If U and V are contractible this restriction map is onto.) $\mathscr{D}(r, n)$ can be identified as the direct limit of the spaces $\mathscr{D}(r, n; U_j)$ where U_j is the ball of radius $1/j$ about 0. We put the direct limit topology on $\mathscr{D}(r, n)$.

One easily checks the following facts regarding these topologies.

(1) The map which assigns to a distribution $\mathscr{D} \in \mathscr{D}(r, n; U)$ its germ $j(\mathscr{D}) \in \mathscr{D}(r, n)$ at 0 is continuous and open. Also the map j^k which assigns to each distribution its k-jet ($k = 0, 1, 2, \ldots$) at 0 is an open continuous projection from $\mathscr{D}(r, n)$ (or $\mathscr{D}(r, n; U)$) onto the space $J^k\mathscr{D}(r, n) = J^k(\mathbf{R}^n, G_{r,n})$ of k-jets at 0.

(2) The group Diff(n) of (germs of) diffeomorphisms which take 0 to 0 (or U to U) acts continuously on $\mathscr{D}(r, n)$ (on $\mathscr{D}(r, n; U)$). The projection j^k intertwines this action with the action of the finite dimensional algebraic group J^{k+1} Diff(n) on $J^k\mathscr{D}(r, n)$. J^{k+1} Diff(n) denotes the group of $k+1$ jets at 0 of diffeomorphisms of $\mathbf{R}^n$ which take 0 to 0.

(3) The spaces $\mathscr{D}(r, n; U)$ are Baire spaces: the intersection of any countable family of open dense sets is dense. It follows that $\mathscr{D}(r, n)$ is also Baire.

Theorem 1 is now an immediate consequence of

PROPOSITION 1. *Let $\Sigma_N \subset \mathscr{D}(r, n)$ denote the set of germs of distributions which admit a frame which generates a Lie algebra of dimension N. Suppose that $r(n-r) > n$. Then the complement of Σ_N contains an open dense set.*

Now recall the notion of the codimension of a subset of $\mathscr{D}(r, n)$. This will allow us to prove and sharpen Proposition 1.

DEFINITION 1. Let $S \subset \mathscr{D}(r, n)$ be a subset of the space of distributions. Suppose that for each k the space $j^k(S)$ of k-jets of S is an algebraic subvariety of $J^k\,\mathscr{D}(r, n)$. If the codimensions of these subvarieties tend to infinity as $k \to \infty$ then we say that S has infinite codimension.

Remark 1. In place of "algebraic" in this definition we could use "subanalytic."

Remark 2. If S is a set of infinite codimension (or even positive finite codimension) then its complement is open and dense. This follows immediately from the facts listed above concerning the Whitney topologies and the fact that the complement of an algebraic subset of positive codimension is open and dense within a finite dimensional manifold.

THEOREM 2. *When $r(n-r)>n$ the orbits $S=\mathrm{Diff}(n)(\mathscr{D})$ of any distribution $\mathscr{D}\in\mathscr{D}(r,n)$ under the action of* Diff(n) *has infinite codimension within $\mathscr{D}(r,n)$. Specifically,* $\mathrm{codim}(j^k(S))\sim r(n-r)(k^n/n!)$ *as k tends to ∞.*

It follows from this theorem and the last remark that in the given dimension range the complement of any orbit in $\mathscr{D}(r,n)$ is open and dense. Consequently, when inequality (1) holds it is impossible for any single distribution to be stable in the usual sense of singularity theory. A *family* of distributions might still be stable but it would have to be an infinite dimensional family modulo Diff(n). The proof of Proposition 1 would be complete if we could show that the families Σ_N are all finite dimensional modulo Diff(n). This is essentially what we do.

In order to do this we use the notion of a cross-section to a group action. For us the group in question is Diff(n).

DEFINITION 2. *Let S be a subset of $\mathscr{D}(r,n)$ which is invariant under the action of* Diff(n). *A "finite dimensional cross-section" to S is a smoothly embedded finite dimensional subvariety $i\colon\mathscr{V}\to\mathscr{D}(r,n)$ whose orbit under* Diff(n) *is all of S:* $\mathrm{Diff}(n)\,i(\mathscr{V})=S$. *Here "smoothly embedded" means that the maps $j^k\circ i\colon\mathscr{V}\to J^k(\mathbf{R}^n,G_{r,n})$ are algebraic embeddings of a real algebraic variety.*

Remark 3. Suppose that S is as in the definition and that $r(n-r)>n$. Then the codimension of S is infinite by Theorem 2. To see this observe that $j^k(S)=J^{k+1}\,\mathrm{Diff}(n)\cdot j^k(i(\mathscr{V}))$ so that $\dim j^k(S)\leqslant\dim(\text{individual orbit})+\dim(\mathscr{V})$. Thus $\mathrm{codim}(j^k(S))\geqslant\mathrm{codim}(\text{individual orbit})-\dim(\mathscr{V})\sim r(n-r)\,k^{n-1}\to\infty$ as $k\to\infty$.

It follows that, given Theorem 2, we will have proved Proposition 1 (and hence Theorem 1) if we can show that each of the sets Σ_N there admits a finite dimensional cross-section. Unfortunately, this is not true in general. However, we will exhibit a dense open subset $\mathscr{D}(r,n)_{gen}$ of $\mathscr{D}(r,n)$ with the property that the set $\Sigma_{N,gen}=\Sigma_N\cap\mathscr{D}(r,n)_{gen}$ does admit such a cross section. Then its complement $\Sigma^c_{N,gen}$ is open and dense. Thus

$$\Sigma^c_N=\Sigma^c_{N,gen}\cap\{\Sigma^c_N\cup\mathscr{D}(r,n)_{gen}\}\subset\Sigma^c_{N,gen}\cap\mathscr{D}(r,n)_{gen}$$

and so is contained in an open dense set, the intersection of two other open dense sets.

DEFINITION 3. *Let $\mathscr{D}$ be a distribution defined in a neighborhood of* 0. *We say $\mathscr{D}$ is bracket generating at* 0 *if it admits a frame E_i, $i=1,2,\ldots,r$, such that the E_i together with their iterated Lie brackets, $[E_i,E_j]$, $[E_i,[E_j,E_k]]$, ... span $\mathbf{R}^n$ upon evaluation at* 0.

$$\mathscr{D}(r,n)_{gen}\subset\mathscr{D}(r,n)$$

will denote the set of all germs of distributions which are bracket generating at 0.

Remark 4. It is easy to check that the bracket generating property is independent of choice of frame. Clearly it is invariant under diffeomorphisms. It is also generic.

PROPOSITION 2. *The set* $\mathscr{D}(r, n)_{gen}$ *consisting of germs of distributions which are bracket generating at* 0 *is an open dense subset of* $\mathscr{D}(r, n)$ *provided* $r > 1$.

From the above discussion it follows that Proposition 1 and hence Theorem 1 will be proved upon proving Theorem 2 and

THEOREM 3. $\Sigma_{N, gen}$ *admits a finite dimensional cross-section.*

4. PROOF OF THEOREM 2

Let S be the orbit of the theorem and set $\mathcal{O}_k = j^k(S)$. According to fact (3) of the previous section $\mathcal{O}_k$ is the orbit of the point $j^k(\mathscr{D}) \in J^k \mathscr{D}(r, n)$ under the action of the (algebraic) group $J^{k+1}\,\mathrm{Diff}(n)$.

The vector space of real-valued polynomials in n variables whose degree is less than or equal to k has dimension equal to the binomial coefficient $\binom{n+k}{k}$. Thus

$$\dim(J^k \mathscr{D}(r, n)) = r(n-r)\binom{n+k}{k}$$

and

$$\dim(J^{k+1}\,\mathrm{Diff}(n)) = n\left(\binom{n+k+1}{k+1} - 1\right).$$

The -1 comes from the fact that the diffeomorphism germs take 0 to 0 and hence the constant term of the polynomials which occur here is zero. It follows that the codimension of the orbit $\mathcal{O}_k$ is *at least*

$$r(n-r)\left(\binom{n+k}{k}\right) - n\left(\binom{n+k+1}{k+1} - 1\right). \tag{2}$$

Now

$$\binom{n+k}{k} \sim \frac{1}{n!}k^n \quad \text{as} \quad k \to \infty. \tag{3}$$

Here $\sim$ has the usual meaning employed in asymptotic expansions:

$$\left[\binom{n+k}{k}-\frac{1}{n!}k^n\right]=\mathcal{O}(k^{n-1}) \qquad \text{as} \quad k\to\infty.$$

Therefore

$$\operatorname{codim} \mathcal{O}_k \sim \frac{1}{n!}\left(r(n-r)\,k^n-n(k+1)^n\right)\sim\left[r(n-r)-n\right]\frac{k^n}{n!}$$

as $k\to\infty$.

5. Proof of Theorem 3

Begin by considering the variety of all Lie algebra structures on $\mathbf{R}^N$. This a finite dimensional real algebraic variety coordinatized by the structure constants $c=c^i_{jk}$ of the Lie algebra. (Thus its dimension is less than N^3.) In addition to this data, consider triples (D_r, W_n, j) where $D_r\subset W_n\subset\mathbf{R}^N$ are linear subspaces whose dimensions are indicated by the subscripts, $j\colon\mathbf{R}^N\to\mathbf{R}^n$ is a linear map with the property that $j(W_n)=\mathbf{R}^n$, and that $\ker(j)$ is a Lie subalgebra of $\mathbf{R}^N$ (with Lie algebra structure as given by the c^i_{jk}), and finally where D_r generates $\mathbf{R}^N$ as a Lie algebra. Denote the set of all such data by $\mathscr{V}_N$. Clearly this forms a real algebraic variety.

We show how to construct the embedding $i\colon\mathscr{V}_N\to\Sigma_N$. First associated to $\tau=(c, D_r, W_n, j)\in\mathscr{V}_N$ the homogeneous space G/H where $G=G(c)$ is the simply connected Lie group with Lie algebra $(\mathbf{R}^N, c)$ and $H=H(j,c)\subset G$ is the closed subgroup whose Lie algebra is $\ker(j)$. G acts on the homogeneous space and so $(\mathbf{R}^N, c)$ acts infinitesimally. That is, for each $v\in\mathbf{R}^N$ we have a vector field $\rho(v)$ on G/H and $[\rho(v),\rho(w)]=\rho(c(v,w))$. Define the distribution $\rho(D_r)$ on G/H to be the one spanned by the vector fields $\rho(v)$, $v\in D_r$. By construction this distribution admits a frame (any basis for D_r) which generates $(\mathbf{R}^N, c)$ as a Lie algebra. (Warning: this distribution is not G invariant.) Our embedding will be complete once we construct a canonical local diffeomorphism $(G/H,[e])\to(\mathbf{R}^n, 0)$. Just use the diffeomorphism to push the distribution on G/H over to $\mathbf{R}^n$. Now we have a canonical chart onto a neighborhood of $[e]$, namely $w\in W_n\mapsto\exp(w)H$. Compose the inverse of this chart with our linear map j to obtain the desired local diffeomorphism.

The composition $j^k\circ i$ is an algebraic embedding. (Truncate the Campbell–Baker–Hausforff formula.)

To see that i is indeed a cross-section, let $\mathscr{D}\in\Sigma_N$. Pick a frame $\{E_i\}$ which generates an N-dimensional Lie algebra $\mathscr{G}$. Choose a linear isomorphism $A\colon\mathbf{R}^N\to\mathscr{G}$ thus inducing a Lie algebra structure c on $\mathbf{R}^N$. If

$e_0 : \mathscr{G} \to \mathbf{R}^n$ is the evaluation map at 0 set $j = L \circ e_0$. $D_r = A(\mathrm{Span}\{E_i\})$. For W_n take any complement to $\ker(j)$ which contains D_r. We have thus formed $\tau = (c, D_r, W_n, j) \in \mathscr{V}_N$. Using the transitvity of the infinitesimal $\mathscr{G}$ action on $\mathbf{R}^n$ and exponential coordinates on $G(c)$ one checks that $\mathscr{D}$ and $i(\tau)$ are locally diffeomorphic distributions.

This completes the proof of Theorem 3 and the paper.

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